

# Fermion Modes in Instanton-Anti-Instanton Background in (1+0)-dimensional Model

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## Abstract

We evaluate the eigenvalues of Dirac operator in the background of instanton-anti-instanton configuration ( $I\bar{I}$ ), in a simple (1+0)-dimensional model, and find that quasi-zero-mode disappears for the closely localized  $I\bar{I}$  configurations. This result suggests that the configurations which are dominant at high energy in the valley method are actually non-anomalous ones and irrelevant to fermion number violation. Hence, there seems to be no theoretical basis for expecting anomalous cross section to become observable at energies of 10 Tev region in Weinberg-Salam theory.

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# 1 Introduction

It is well known that in Weinberg-Salam theory baryon number is violated due to anomaly non-perturbatively, but a naive semiclassical approximation shows that such processes are strongly suppressed by the factor of  $e^{-2S_I}$  where  $S_I$  is the instanton action [1]. These processes had been therefore considered as hopelessly unobservable until, in 1990, Ringwald [2] and Espinosa [3] suggested that the cross section grows exponentially at high energy when  $O(1/\alpha)$  gauge and Higgs bosons in the final states are considered, because of the increase in available phase space. Since then, a lot of modifications of their calculations have been made [4], trying to solve the unitarity bound problem and to apply them to the case at the energy above the Sphaleron.

Among them, the valley method [5, 6, 7, 8, 9] showed an optimistic result suggesting that the anomalous cross section reaches an observable one without breaking the unitarity bound at the energy of the order of  $E_{sp}$  ( $\sim 10\text{TeV}$ ) [8]. In this method, total cross section  $\sigma_{tot}$  in the instanton background (I) is computed, via optical theorem, as the imaginary part of the forward elastic scattering amplitude ( $\Im(FES)$ ) in the instanton-anti-instanton background(I- $\bar{I}$ ),

$$\sigma_{tot,I} = \Im(FES_{I-\bar{I}}), \quad (1)$$

so that the final state corrections are automatically included.

However, since we are interested in the "anomalous" processes where baryon number is violated, we have to extract the correct part from the I- $\bar{I}$  configurations, *i.e.*, fermion number of the intermediate state in the background of I- $\bar{I}$  configuration must be topologically correct, and not just the same as that of the initial (and final) state (Fig. 1). When the initial state energy is increased, the configurations where the separation between instanton and anti-instanton is much smaller than the instanton size become dominant [7, 8], but such configurations seem to be non-anomalous. Therefore the anomalous part may be only a tiny fraction of their optimistic estimate of the cross section [10, 11].

Although both of the perturbative vacuum and the far separated I- $\bar{I}$  configuration have zero instanton number globally, there should be some (qualitative) distinction between them. In this paper, we consider a simple (1+0)-dimensional model, and investigate the quasi-zero-mode of Dirac operator.

## 2 (1+0)-dimensional Toy Model

In this section, we introduce a toy model which has the same features as the Weinberg-Salam model, *i.e.*, instantons and anomaly.<sup>1</sup> Closely related models have been used in a variety of contexts : Witten's SUSY quantum mechanics [12] and fermion number fractionization in polymer physics [13]. In all of these past applications, they considered quantum mechanics in 1-spatial dimension (*i.e.*, (1+1)-dimensional field theory), whereas we use this model as a (1+0)-dimensional field theory, and consequently physical contents are completely different.

This model has a boson field  $\Phi(t)$  in a double-well-potential and a two-component fermion field  $\Psi(t)$  coupled to the boson  $\Phi(t)$  by Yukawa-type interaction. The Euclidean action and the associated Hamiltonian read as follows.

$$S_E = \int dt [\bar{\Psi} D \Psi + \frac{1}{2} \dot{\Phi}^2 + V(\Phi)], \quad (2)$$

$$\begin{aligned} D &= -i\sigma_2 \partial_t + \sigma_1 \Phi \\ &= \begin{pmatrix} 0 & -\partial_t + \Phi \\ \partial_t + \Phi & 0 \end{pmatrix}, \end{aligned} \quad (3)$$

$$H = \Phi \Psi^\dagger \sigma_3 \Psi + \frac{1}{2} p^2 + V(\Phi), \quad (4)$$

where  $\sigma_i$ 's are Pauli-matrices, and  $V(\Phi)$  is the double well potential.

First, let's consider the energy-spectrum of fermion when a background configuration  $\Phi$  is at the bottoms of the wells (Fig. 2(a)). Since the Hamiltonian for fermion has the form in Eq. (4), the up (down)-component has positive (negative) energy in the right well and negative (positive) energy in the left, respectively (Fig. 2(b)). Therefore, when the background configuration  $\Phi$  is changed from the left well to the right well, the fermion number of up (down) component increases by 1 ( $-1$ ).

Next, let's consider the zero-mode of (1+0)-dimensional Dirac operator  $D$  in the instanton (or anti-instanton) background. The equations of zero-mode for up and down components

$$(\partial_t + \Phi) \Psi_u = 0, \quad (5)$$

$$(-\partial_t + \Phi) \Psi_d = 0 \quad (6)$$

can be easily integrated to give

$$\Psi_u = \exp(-\int dt \Phi), \quad (7)$$

$$\Psi_d = \exp(\int dt \Phi). \quad (8)$$

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<sup>1</sup>Although it doesn't appear to be possible to derive local anomaly (non-vanishing current divergence) since it is a one dimensional model, it has global anomaly as shown below.

When the background  $\Phi$  is an instanton, *i.e.*

$$\text{Index} \equiv \frac{1}{2}[\text{sign}\Phi(t = \infty) - \text{sign}\Phi(t = -\infty)] \quad (9)$$

$$= +1, \quad (10)$$

$\Psi_u$  is normalizable, but  $\Psi_d$  is not. In the case of anti-instanton the above situations are reversed. Since  $D$  is a hermitian operator,  $\Psi_u$  and  $\bar{\Psi}_d$  have a zero-mode in the instanton background, while in the anti-instanton background  $\Psi_d$  and  $\bar{\Psi}_u$  have a zero-mode. Therefore the following Green's functions give non-vanishing results, in the instanton and anti-instanton background, respectively,

$$\langle 0|T(\Psi_u(t_1)\bar{\Psi}_d(t_2))|0\rangle \neq 0 \quad (\text{instanton}), \quad (11)$$

$$\langle 0|T(\Psi_d(t_1)\bar{\Psi}_u(t_2))|0\rangle \neq 0 \quad (\text{anti-instanton}). \quad (12)$$

Consequently, the fermion numbers change as follows,

$$\Delta N_u = -\Delta N_d = 1 \quad (\text{instanton}), \quad (13)$$

$$\Delta N_u = -\Delta N_d = -1 \quad (\text{anti-instanton}), \quad (14)$$

which gives the same results derived above from the energy-spectrum consideration.

Finally, let's see the correspondence of non-zero-modes between up and down components. Dirac operator satisfies the following relations

$$\begin{aligned} \{D, \sigma_3\} &= 0, \\ [D^2, \sigma_3] &= 0, \end{aligned} \quad (15)$$

where

$$\begin{aligned} D &= -i\sigma_2\partial_t + \sigma_1\Phi = \begin{pmatrix} 0 & -\partial_t + \Phi \\ \partial_t + \Phi & 0 \end{pmatrix}, \\ D^2 &= \begin{pmatrix} -\partial_t^2 + U_u & 0 \\ 0 & -\partial_t^2 + U_d \end{pmatrix}, \\ U_u &= \Phi^2 - \partial_t\Phi, \quad U_d = \Phi^2 + \partial_t\Phi. \end{aligned} \quad (16)$$

Consequently, the eigenstates for both  $D^2$  and  $\sigma_3$

$$D^2\Psi_u = \lambda^2\Psi_u, \quad \sigma_3\Psi_u = +\Psi_u, \quad (17)$$

$$D^2\Psi_d = \lambda^2\Psi_d, \quad \sigma_3\Psi_d = -\Psi_d, \quad (18)$$

have the following one to one correspondence for  $\lambda \neq 0$ ,

$$\Psi_d = \frac{1}{\lambda}D\Psi_u, \quad (19)$$

$$\Psi_u = \frac{1}{\lambda}D\Psi_d. \quad (20)$$

Therefore, we can obtain the eigenstates of  $D$  from these eigenstates of  $D^2$  and  $\sigma_3$  in the following way,

$$D \left[ \frac{\Psi_u \pm \Psi_d}{\sqrt{2}} \right] = \pm \lambda \left[ \frac{\Psi_u \pm \Psi_d}{\sqrt{2}} \right]. \quad (21)$$

### 3 Quasi-Zero-Mode in I- $\bar{I}$ Background

Using the (1+0)-dimensional toy model presented in section 2, we calculate numerically the spectrum of  $D$  in the background of I- $\bar{I}$  configurations,

$$\Phi(t) = v \left[ \tanh\left(\frac{1}{\rho}\left(t + \frac{1}{2}R\right)\right) - \tanh\left(\frac{1}{\rho}\left(t - \frac{1}{2}R\right)\right) - 1 \right], \quad (22)$$

where  $v$  and  $\rho$  are the vacuum expectation value of  $\Phi$  and the instanton size, respectively, which are determined by the double well potential;  $R$  is the I- $\bar{I}$  separation (Fig. 3).

Fig. 4 shows the smallest eigenvalue of  $D^2$ , and Fig. 5 to Fig. 7 show their eigenfunctions. These figures show that there exists a quasi-zero-mode whose eigenfunctions are localized at the position of instanton or anti-instanton for a large instanton-anti-instanton separation  $R$ ; this behavior suggests the existence of the topologically correct fermions in the intermediate state. On the contrary, when the distance divided by the instanton size,  $R/\rho$ , becomes of the order of unity, the quasi-zero-mode disappears and the transition to no-bound continuous mode takes place; the intermediate state is no longer an anomalous one.

However, these figures also suggest that there is no definite value of  $R/\rho$  which distinguishes the anomalous configurations from the non-anomalous ones. Indeed the value of  $R/\rho$  where the smallest eigen value departs from zero appreciably, depends on the parameter  $v$ : the extension of the eigenfunctions is of the order of  $1/v$ , and when they get close to overlap each other, the smallest eigenvalue starts to increase.

## 4 Conclusion and Discussion

We obtain a clear distinction between the perturbative vacuum and the far separated instanton-anti-instanton configuration, when considered as intermediate states in the valley method: When instanton and anti-instanton get close and their separation becomes of the order of instanton size, the quasi-zero-mode of Dirac operator disappears.

Seen from Hamiltonian picture, we can obtain the same result. The Hamiltonian for the up(down)-component fermion is just the background  $\Phi$  ( $-\Phi$ ) in this model (Eq. (4)), and we can also see from Eq. (9) that Chern-Simons number is also the value of background  $\Phi$ , essentially. Consequently, it can be easily seen that the flow

of energy-spectrum crosses the zero value twice in the case of far separated instanton-anti-instanton, whereas below the critical value  $R_c/\rho = 2\text{arctanh}(1/2) \sim 1.1$ , the level crossings are absent.

Although we can obtain the clear results easily from the model which we have presented in this paper, this toy model is a much simplified (1+0)-dimensional one, so some features could be different from those of a realistic 4-dimensional SU(2) gauge theory. For instance, in Ref. [10] they claim that there's no quasi-zero-mode at all, nor is there any mode related to the fermion number violation for I- $\bar{\text{I}}$  background, in 4-dimensional gauge theory.

Nevertheless, since only the far separated instanton-anti-instanton configurations which have correct topology in the intermediate states are relevant for the baryon number violation, the optimistic result based on the valley method seems to be invalid in the present toy model, also.

The meaning of quasi-zero-mode in the Dirac operator and a precise analysis of fermion number in the intermediate states will be considered diagrammatically elsewhere [14].

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## Figure Captions

Fig. 1. Correct behavior of intermediate state fermions in the anomalous processes. At each vertex of instanton (I) and anti-instanton ( $\bar{I}$ ), fermion number have to change correctly.

Fig. 2. Double-well-potential for boson (a), and energy-spectrum of fermions (b).

Fig. 3. Instanton-anti-instanton configuration.

Fig. 4. The smallest eigenvalue  $\lambda^2$  of  $D^2$  versus  $R/\rho$ , where  $R$  is the I- $\bar{I}$  separation and  $\rho$  is the instanton size. The vacuum expectation value of  $\Phi$  at the bottoms of the wells,  $v$ , is set to be  $1/(v\rho) = 2.5$  in this case. When  $R/\rho$  becomes of the order of unity, quasi-zero-mode changes to continuous mode. Note that continuous modes start from  $\lambda^2 \sim v^2$ .

Fig. 5. Wave function for  $R/\rho = 15$ .

Dashed line shows background  $\Phi$ , the straight line  $\Psi_u$ , and the dot line  $\Psi_d$ , respectively. When instanton and anti-instanton are far separated, wave functions of fermion are bounded at the instanton (or anti-instanton) position.

Fig. 6. Wave function for  $R/\rho = 1.1$ .

Wave functions are overlapping.

Fig. 7. Wave function for  $R/\rho = 0.1$ .

Dot-dashed line shows the wavefunction in the case of perturbative-vacuum. Transition from bound state to no-bound continuous mode takes place.



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